UNIT - IV Riemann Integration

$F \in R[a,b]$ means F is Riemann integral on $[a,b]$

Definition: let $[a,b]$ be a closed interval then the finite set $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ is called partition of a closed interval $[a,b]$

Formulas :

1) The $(n+1)$ points $x_0, x_1 \dots x_n$ are called partition points of P.
2) The 'n' subintervals $[x_0, x_1], [x_1, x_2] \dots [x_{n-1}, x_n]$ are called segments of P.
3) The rth sub interval $[x_{r-1}, x_r]$ is denoted by I_r.
4) The length of I_r is $\delta_r = x_r - x_{r-1}$
5) The set of all partitions of $[a,b]$ is denoted by $\wp[a,b]$.
6) If P_1, P_2 are two partitions of $[a,b]$ then P_1 and P is contained in c then we say that P_2 is a refinement of P_1.
7) If P_1, P_2 are two partitions of $[a,b]$ then $P_1 \cup P_2$ is a common refinment of P_1 and P_2.
8)
$$\sum_{r=1}^{n} \delta_r = \delta_1 + \delta_2 + \dots + \delta_n$$
$$= x_1 - x_0 + x_2 - x_1 + \dots + x_n - x_{n-1}$$
$$= x_n - x_0$$
$$= b - a$$
9) 'f' is bounded on $[a,b]$ then 'f' is bounded on each subinterval of P.
10) Let M_r, m_r be the superimum and infimum of 'f' on I_r, Where $r = 1, 2 \dots n$ then

i) $\sum_{r=1}^{n} M_r \delta_r$ is called upper Riemann sum. It is denoted by $U(P,F)$ $\therefore U(P,F) = \sum_{r=1}^{n} M_r \delta_r$

ii) $\sum_{i=1}^{n} m_r \delta_r$ is called lower Riemann sum. It is denoted by $L(P,F)$.

$$\therefore L(P,F) = \sum_{r=1}^{n} m_r \delta_r.$$

ii) <u>Normal of P</u>:

The maximum length of the subinterval of 'p' is called Norm of P. It is denoted by $\|P\|$.

<u>Theorem ①</u>:

If $f: [a,b] \rightarrow R$ is bounded function and $D \in \emptyset [a,b]$ then

i) $U(P,f) \geq L(P,f)$

ii) $U(P,-f) = -L(P,f)$

iii) $L(P,-f) = -U(P,f)$

<u>Proof</u>: Let $P = \{a = x_0, x_1, \ldots x_n = b\}$ be the partition of $[a,b]$

Let M_r, m_r be the sup and Inf of 'f' on $[a,b]$

Then M_r, m_r be the sup and Inf of 'f' on I_r.

i) We have $M_r \geq m_r$

$$M_r \delta_r \geq m_r \delta_r$$

$$\sum_{r=1}^{n} M_r \delta_r \geq \sum_{r=1}^{n} m_r \delta_r$$

$$U(P,f) \geq L(P,f)$$

ii) Since M_r, m_r be the sup and Inf of 'f' on I_r. $-m_r$ and $-M_r$ be the sup and Inf.

$$\text{Now, } U(P,-f) = \sum_{r=1}^{n} -m_r \delta_r$$

$$= -\sum_{r=1}^{n} m_r \delta_r$$

$$U(P,-f) = -L(P,f)$$

iii) $$L(P,-f) = \sum_{r=1}^{n} -M_r \delta_r$$

$$= -\sum_{r=1}^{n} M_r \delta_r$$

$$L(P,-f) = -U(P,f)$$

Oscillatory sum:

If $f:[a,b] \to R$ is a bounded function then $U(P,f) - L(P,f)$ is called the oscillatory sum.
It is denoted by $W(P,f)$ or $\omega(P,f)$

$$\therefore W(P,f) = U(P,f) - L(P,f)$$

Theorem 2:

If $f:[a,b] \to R$ is a bounded function and $P_1, P_2 \in \wp[a,b]$ such that $P_1 \subset P_2$

i) $U(P_1, f) \geq U(P_2, f)$

ii) $L(P_1, f) \leq L(P_2, f)$

iii) $W(P_1, f) \geq W(P_2, f)$

Proof: suppose $f:[a,b] \to R$ is a bounded function and $P_1, P_2 \in \wp[a,b]$ show that $P_1 \subset P_2$

suppose P_2 contains only one element more than P_1.

$P_1 = \{a = x_0, x_1, x_2, \ldots\ldots x_{r-1}, x_r \ldots\ldots x_n = b\}$

$P_2 = \{a = x_0, x_1, x_2, \ldots\ldots x_{r-1}, y, x_r \ldots\ldots x_n = b\}$

$M_r\, m_r$ be the sup and Inf of 'f' on I_r
$= [x_{r-1}, x_r]$

λ_1, λ_2 be the sup of 'f' on $[x_{r-1}, y]$, $[y, x_r]$ respectively.

μ_1, μ_2 be the sup of 'f' on $[x_{r-1}, y]$, $[y, x_r]$ respectively.

$\therefore M_r \geq \lambda_1, \lambda_2$
$m_r \leq \mu_1, \mu_2$

i) $U(P_1, f) - U(P_2, f) = \{M_1 \delta_1 + M_2 \delta_2 + \dots + M_{r-1} \delta_{r-1} + M_r \delta_r + \dots + M_n \delta_n\} - \{M_1 \delta_1 + M_2 \delta_2 + \dots + M_{r-1} \delta_{r-1} + \lambda_1 (y - x_{r-1}) + \lambda_2 (x_r - y) + \dots + m_1 \delta_n\}$.

$\Rightarrow M_r \delta_r - \{\lambda_1 (y - x_r - 1) + \lambda_2 (x_r - y)\}$

$\Rightarrow M_r (x_r - x_r - 1) - \lambda_1 (y - x_r - 1) - \lambda_2 (x_r - y)$

$\Rightarrow M_r (x_r - y + y - x_r - 1) - \lambda_1 (y - x_r - 1) - \lambda_2 (x_r - y)$

$= M_r (x_r - y) + M_r (y - x_{r-1}) - \lambda_1 (y - x_r - 1) - \lambda_2 (x_r - y)$

$= (x_r - y)(M_r - \lambda_2) + (y - x_r - 1)(M_r - \lambda_1) \geq 0$

$U(P_1, f) - U(P_2, f) \geq 0$

$\therefore U(P_1, f) \geq U(P_2, f)$ $\quad (M_r \geq \lambda_1, \lambda_2)$

ii) $L(P_1, F) - L(P_2, F) = \{m_1 \delta_1 + m_2 \delta_2 + \dots + m_{r-1} \delta_{r-1} + m_r \delta_r + \dots m_n \delta_n\} - \{m_1 \delta_1 + m_2 \delta_2 + \dots + m_{r-1} \delta_{r-1} + \mu_1 (y - x_{r-1}) + \mu_2 (x_r - y) + \dots + m_n \delta_n\}$

$= m_r \delta_r - \{\mu_1 (y - x_r - 1) + \mu_2 (x_r - y)\}$

$= m_r (x_r - x_r - 1) - \mu_1 (y - x_{r-1}) - \mu_2 (x_r - y)$

$= m_r (x_r - y + y - x_{r-1}) - \mu_1 (y - x_{r-1}) - \mu_2 (x_r - y)$

$= m_r (x_r - y) + m_r (y - x_r - 1) - \mu_1 (\mu - x_r - 1) - \mu_2 (x_r - y)$

$= (x_r - y)(m_r - \mu_2) + (y - x_r - 1)(m_r - \mu_1) \leq 0$ $\quad (m_r \leq \mu_1, \mu_2)$

$L(P_1, f) - L(P_2, f) \leq 0$

$\therefore L(P_1, f) \leq L(P_2, f)$

iii) From (i) & (ii) $\quad U(P_1, f) \geq U(P_2, f)$

$L(P_1, f) \leq L(P_2, f)$

$-L(P_1, f) \geq -L(P_2, f)$

$$U(P_1, f) - L(P_1, f) \geq U(P_2, f) - L(P_2, f)$$

$$W(P_1, f) \geq W(P_2, f)$$

Hence proved.

Lower Riemann Integral:

If $f : [a,b] \to R$ be a bounded function then sup $\{L(P,f) / P \in \wp[a,b]\}$ is called lower Riemann Integral of 'f' on $[a,b]$. It is denoted by $\underline{\int_a^b} f(x)\,dx$

$$\therefore \underline{\int_a^b} f(x)\,dx = \sup \{L(P,f) / P \in \wp[a,b]\}$$

Upper Riemann Integral:

If $f : [a,b] \to R$ be a bounded function then inf $\{U(P,f) / P \in \wp[a,b]\}$ is called URI of 'f' on $[a,b]$. It is denoted by $\overline{\int_a^b} f(x)\,dx$

$$\therefore \overline{\int_a^b} f(x)\,dx = \inf \{U(P,f) / P \in \wp[a,b]\}$$

Note:

i) $\underline{\int_a^b} f(x)\,dx \leq \mu(b-a)$

ii) $\overline{\int_a^b} f(x)\,dx \geq \mu(b-a)$

Riemann Integral Function:

If $f : [a,b] \to R$ be a bounded function then we say that 'f' is Riemann Integral Function if $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$.

It is denoted by $\int_a^b f(x)\,dx$.

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

Problems:

1) If $f(x) = x$ on $[0,1]$ and $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Find $U(P,f)$ and $L(P,f)$

Sol: Given $f(x) = x$ on $[0,1]$ & $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

$I_1 = [0, \frac{1}{3}]$, $I_2 = [\frac{1}{3}, \frac{2}{3}]$, $I_3 = [\frac{2}{3}, 1]$.

$\delta = \frac{1}{3} - 0 = \frac{1}{3}$ | $\delta_2 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ | $\delta_3 = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore f(x)$ is increasing Function

M_1 is sup of 'f' on $I_1 = \frac{1}{3}$

M_2 is sup of 'f' on $I_2 = \frac{2}{3}$

M_3 is sup of 'f' on $I_3 = 1$

$$U(P,f) = \sum_{r=1}^{3} M_r \delta_r$$

$$= M_1\delta_1 + M_2\delta_2 + M_3\delta_3$$

$$= (\tfrac{1}{3})(\tfrac{1}{3}) + (\tfrac{2}{3})(\tfrac{1}{3}) + 1(\tfrac{1}{3})$$

$$= \frac{1}{9} + \frac{2}{9} + \frac{1}{3}$$

$$= \frac{1+2+3}{9} = \frac{6}{9} = \frac{2}{3}.$$

$$U(P,f) = \frac{2}{3}$$

m_1 is inf of 'f' on $I_1 = 0$

m_2 is inf of 'f' on $I_2 = \frac{1}{3}$

m_3 is inf of 'f' on $I_3 = \frac{2}{3}$

$$L(P,f) = \sum_{r=1}^{3} m_r \delta_r$$

$$= m_1\delta_1 + m_2\delta_2 + m_3\delta_3$$

$$= 0 + (\tfrac{1}{3})(\tfrac{1}{3}) + (\tfrac{2}{3})(\tfrac{1}{3})$$

$$= 0 + \frac{1}{9} + \frac{2}{9}$$

$$= \frac{1+2}{9}$$

$$= \frac{3}{9} = \frac{1}{3} \qquad L(P,f) = \frac{1}{3}$$

2) If $f(x) = x^2$ on $[0,1]$ and $P = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$. Find $U(P,f)$ and $L(P,f)$

sol: Given $f(x) = x^2$ on $[0,1]$ & $P = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$

$I_1 = [0, \frac{1}{4}]$, $I_2 = [\frac{1}{4}, \frac{2}{4}]$, $I_3 = [\frac{2}{3}, \frac{3}{4}]$, $I_4 = [\frac{3}{4}, 1]$

$\delta_1 = \frac{1}{4} - 0$ $\quad \delta_2 = \frac{2}{4} - \frac{1}{4}$ $\quad \delta_3 = \frac{3}{4} - \frac{2}{4}$ $\quad \delta_4 = 1 - \frac{3}{4}$

$= \frac{1}{4}$ $\quad = \frac{1}{4}$ $\quad = \frac{1}{4}$ $\quad = \frac{1}{4}$

$\therefore f(x)$ is increasing function

M_1 is sup of 'f' on $I_1 = (\frac{1}{4})^2 = \frac{1}{16}$

M_2 is sup of 'f' on $I_2 = (\frac{2}{4})^2 = \frac{4}{16} = \frac{1}{4}$

M_3 is sup of 'f' on $I_3 = (\frac{3}{4})^2 = \frac{9}{16}$

M_4 is sup of 'f' on $I_4 = (1)^2 = 1$

$$U(P,f) = \sum_{i=1}^{4} M_i \delta_i$$

$$= M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4$$

$$= (\frac{1}{16})(\frac{1}{4}) + (\frac{4}{16})(\frac{1}{4}) + (\frac{9}{16})(\frac{1}{4}) + 1(\frac{1}{4})$$

$$= \frac{1}{64} + \frac{4}{64} + \frac{9}{64} + \frac{1}{4}$$

$$= \frac{1+4+9+16}{64} = \frac{30}{64} = \frac{15}{32}$$

m_1 is inf of 'f' on $I_1 = (0)^2 = 0$

m_2 is inf of 'f' on $I_2 = (\frac{1}{4})^2 = \frac{1}{16}$

m_3 is inf of 'f' on $I_3 = (\frac{2}{4})^2 = \frac{4}{16}$

m_4 is inf of 'f' on $I_4 = (\frac{3}{4})^2 = \frac{9}{16}$

$$L(P,f) = \sum_{r=1}^{4} m_r \delta_r$$
$$= m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + m_4\delta_4$$
$$= 0\left(\frac{1}{4}\right) + \frac{1}{16}\left(\frac{1}{4}\right) + \frac{4}{16}\left(\frac{1}{4}\right) + \frac{9}{16}\left(\frac{1}{4}\right)$$
$$= 0 + \frac{1}{64} + \frac{4}{64} + \frac{9}{64}$$
$$= \frac{14}{64} = \frac{7}{32}$$

3) If $f(x) = 2x-1$ on $[0,1]$ and $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$. Find $U(P,f)$ & $L(P,f)$

sol: Given $f(x) = 2x$ on $[0,1]$ & $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$

$I_1 = \left[0, \frac{1}{3}\right]$, $I_2 = \left[\frac{1}{3}, \frac{2}{3}\right]$, $I_3 = \left[\frac{2}{3}, 1\right]$

$\delta_1 = \frac{1}{3} - 0 = \frac{1}{3}$ $\quad \delta_2 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ $\quad \delta_3 = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore f(x)$ is increasing Function

M_1 is sup of 'f' on $I_1 = 2\left(\frac{1}{3}\right) - 1 = \frac{2}{3} - 1 = \frac{-1}{3}$

M_2 is sup of 'f' on $I_2 = 2\left(\frac{2}{3}\right) - 1 = \frac{4}{3} - 1 = \frac{1}{3}$

M_3 is sup of 'f' on $I_3 = 2(1) - 1 = 2 - 1 = 1$

$$U(P,f) = \sum_{r=1}^{3} M_r \delta_r$$
$$= M_1\delta_1 + M_2\delta_2 + M_3\delta_3$$
$$= \left(\frac{-1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right)$$
$$= \frac{-1}{9} + \frac{1}{9} + \frac{1}{3}$$
$$= \frac{1}{3}$$

m_1 is inf of 'f' on $I_1 = 2(0) - 1 = -1$

m_2 is inf of 'f' on $I_2 = 2\left(\frac{1}{3}\right) - 1 = \frac{2}{3} - 1 = \frac{-1}{3}$

m_3 is inf of 'f' on $I_3 = 2\left(\frac{2}{3}\right) - 1 = \frac{4}{3} - 1 = \frac{1}{3}$

$$L(P,f) = \sum_{r=1}^{3} m_r \delta_r$$
$$= m_1\delta_1 + m_2\delta_2 + m_3\delta_3$$

$= (-1)\left(\frac{1}{3}\right) + \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$

$= \frac{-1}{3} - \frac{1}{9} + \frac{1}{9}$

$= \frac{-3-1+9}{9}$

$= \frac{-3}{9} = \frac{-1}{3}$

4) If $f(x) = \sin x$ on $[0, \pi]$ and $P = \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$. Find $U(P,f)$ and $L(P,f)$

Sol: Given $f(x) = \sin x$ on $[0, \pi]$ & $P = \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$

$I_1 = \left[0, \frac{\pi}{3}\right]$, $I_2 = \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$, $I_3 = \left[\frac{2\pi}{3}, \pi\right]$

$\delta_1 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$, $\delta_2 = \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$, $\delta_3 = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

$\therefore f(x)$ is ~~an~~ increasing function.

M_1 is sup of 'f' on $I_1 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

M_2 is sup of 'f' on $I_2 = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

M_3 is sup of 'f' on $I_3 = \sin\pi = 0$

$U(P,f) = \sum_{r=1}^{3} M_r \delta_r$

$= M_1\delta_1 + M_2\delta_2 + M_3\delta_3$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{3}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{3}\right) + 0\left(\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}\pi}{6} + \frac{\sqrt{3}\pi}{6} = \frac{2\sqrt{3}\pi}{6} = \frac{\sqrt{3}\pi}{3}$

$= \frac{\cancel{\sqrt{3}}\pi}{\cancel{\sqrt{3}}\sqrt{3}} = \frac{\pi}{\sqrt{3}}$

m_1 is inf of 'f' on $I_1 = \sin 0 = 0$

m_2 is inf of 'f' on $I_2 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

m_3 is inf of 'f' on $I_3 = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$$L(P,f) = \sum_{r=1}^{3} m_r \delta_r$$

$$= m_1\delta_1 + m_2\delta_2 + m_3\delta_3.$$

$$= 0\left(\frac{\pi}{3}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{3}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}\pi + \sqrt{3}\pi}{6} = \frac{\cancel{2}\sqrt{3}\pi}{\cancel{6}\,3}$$

$$= \frac{\sqrt{3}\pi}{\sqrt{3}\sqrt{3}}$$

$$= \frac{\pi}{\sqrt{3}}$$

5) The Function $f(x) = 1$, if $x \in Q$ & $f(x) = -1$ if $x \in R - \phi$ is not integrable on $[a,b]$?

Sol: Given $f(x) = \begin{cases} 1, & x \in Q \\ -1, & x \in R-Q \end{cases}$

Let $P = \{a = x_1, x_2 \ldots x_n \ldots x_n = b\}$ be a partisian on $[a,b]$ let M_r, m_r be the sup and inf of f on $I_r = [x_{r-1}, x_r]$

Let $M_r = 1$ and $m_r = -1$

now $$U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= \sum_{r=1}^{n} 1.\delta_r = \sum_{r=1}^{n} \delta_r$$

$$U(P,f) = b.a$$

$$\text{Inf}\{U(P,f) \mid P \in \phi\ [a,b]\} = b-a$$

$$\int_a^b f(x)\,dx = b-a \rightarrow (1)$$

Now $$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= \sum_{r=1}^{n} (-1)\,\delta_r \Rightarrow \sum_{r=1}^{n} -\delta_r$$

$L(P,f) = -(b-a)$

$\sup\{L(P,f) \mid P \in \wp[a,b]\} = -(b-a)$

$\underline{\int_a^b} f(x)\,dx = -(b-a) \rightarrow ②$

From ① & ② We get

$\overline{\int_a^b} f(x)\,dx \neq \underline{\int_a^b} f(x)\,dx$

$f(x)$ is Reimann integral on $[a,b]$

NOTE: $\overline{\int_a^b} f(x)\,dx = \underset{n\to\infty}{Lt}\, U(P,f)$

$\underline{\int_a^b} f(x)\,dx = \underset{n\to\infty}{Lt}\, L(P,f)$

Imp

6) S.T $f(x) = 3x+1$ is integrable on $[1,2]$ and $\int_1^2 f(x)\,dx = {}^{11}/_{2}$

sol: Given $f(x) = 3x+1$ on $[1,2]$

$\therefore f(x)$ is bounded on $[1,2]$

Let $P = \left\{ a, a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}, \dots a+\frac{r(b-a)}{n} \dots\dots a+\frac{n(b-a)}{n} \right\}$

Now, $P = \left\{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \dots 1+\frac{r}{n} \dots 1+\frac{n}{n}\right\}$ on $[1,2]$

Let $I_r = \left[1+\frac{r-1}{n}, 1+\frac{r}{n}\right]$

$\delta_r = 1+\frac{r}{n} - \left[1+\frac{r-1}{n}\right]$

$\Rightarrow \cancel{1}+\cancel{\frac{r}{n}} - \cancel{1} - \cancel{\frac{r}{n}} + \frac{1}{n}$

$\Rightarrow \frac{1}{n}$

$\therefore F$ is increasing Function

$$F\left(1+\frac{r}{n}\right) = 3\left(1+\frac{r}{n}\right)+1$$

$$= 3+\frac{3r}{n}+1$$

$$= 4+\frac{3r}{n}$$

$$\therefore M_r = 4+\frac{3r}{n}$$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= \sum_{r=1}^{n}\left(4+\frac{3r}{n}\right)\left(\frac{1}{n}\right)$$

$$= \sum_{r=1}^{n}\left(\frac{4}{n}+\frac{3r}{n}\right)$$

$$= \frac{4}{n}\sum_{r=1}^{n}(1) + \frac{3}{n^2}\sum_{r=1}^{n}(r)$$

$$= \frac{4}{n}(n) + \frac{3}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 4+\frac{3}{2\cancel{n}}\left[\cancel{n}\left(1+\frac{1}{n}\right)\right]$$

$$= 4+\frac{3}{2}\left[1+\frac{1}{n}\right]$$

now $\int_1^2 f(x)\,dx = \underset{n\to\infty}{Lt}\; U(P,f)$

$$= \underset{n\to\infty}{Lt}\; 4+\frac{3}{2}\left[1+\frac{1}{n}\right]$$

$$\int_1^2 f(x)\,dx = 4+\frac{3}{2}\left(1+\frac{1}{\infty}\right)$$

$$= 4+\frac{3}{2}$$

$$= \frac{8+3}{2} = \frac{11}{2} \rightarrow 1$$

m_r be inf of 'f' on $I_r = 1+\frac{r-1}{n}$

$$f\left[\frac{1+r-1}{n}\right] = 3\left[1+\frac{r-1}{n}\right]+1$$

$$m_r = 3+\frac{3r}{n}-\frac{3}{n}+1$$

$= 4 + \frac{3(r-1)}{n}$

$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$= \sum_{r=1}^{n} \left[4 + \frac{3(r-1)}{n} \cdot \frac{1}{n}\right]$

$= \sum_{r=1}^{n} \left[\frac{4}{n} + \frac{3(r-1)}{n}\right]$

$= \frac{4}{n} \sum_{r=1}^{n} (1) + \frac{3}{n^2} \sum_{r=1}^{n} (r-1)$

$= \frac{4}{n} \times n + \frac{3}{n^2} \times \frac{n(n-1)}{2}$

$= 4 + \frac{3}{2}\left(1 - \frac{1}{n}\right)$

$\int_1^2 f(x)\,dx = \lim_{n\to\infty} L(P,f)$

$= \lim_{n\to\infty} \left[4 + \frac{3}{2}\left(1 - \frac{1}{n}\right)\right]$

$= 4 + \frac{3}{2}\left(1 - \frac{1}{\infty}\right)$

$= \frac{8+3}{2} = \frac{11}{2}$

$\overline{\int_1^2} f(x)\,dx = \int_1^2 f(x)\,dx = \frac{11}{2}$

$\int_1^2 f(x)\,dx = \frac{11}{2}$

7) S.T $f(x) = x^2$, ~~on~~ $[0,a]$ Riemman integral on $[0,a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$

Sol: Given $f(x) = x^2$, on $[0,a]$

Let $P = \{a, 0 + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots\ldots a + \frac{r(b-a)}{n} + \ldots a + \frac{n(b-a)}{n}\}$

Now $P = \{0, a + \frac{a-0}{n}, 0 + \frac{2(a-0)}{n}, \ldots\ldots 0 + \frac{r(a-0)}{n}, \ldots\ldots 0 + \frac{n(a-0)}{n}\}$

let $I_r = \left(\frac{(r-1)a}{n}, \frac{ra}{n}\right)$

$\delta_r = \frac{ra}{n} - \frac{ra}{n} + \frac{a}{n} \Rightarrow \frac{a}{n}$

$\therefore f$ is increasing Function on $[0,a]$

M_r be the sup of 'f' on $I_r = \frac{ra}{n}$

$f\left(\frac{ra}{n}\right) = \left(\frac{ra}{n}\right)^2$

$M_r = \left(\frac{ra}{n}\right)^2$

$\therefore U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$= \sum_{r=1}^{n} \left(\frac{ra}{n}\right)^2 \frac{a}{n}$

$= \sum_{r=1}^{n} \frac{r^2 a^3}{n^3}$

$= \frac{a^3}{n^3} \sum_{r=1}^{n} (r)^2$

$= \frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6}$

$= \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

Now $\int_0^a f(x)\,dx = \underset{n\to\infty}{Lt}\ U(P,f)$

$= \underset{n\to\infty}{Lt} \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$= \frac{a^3}{6}(1+0)(2+0)$

$= \frac{a^3}{6} \times 2 \Rightarrow \frac{a^3}{3} \rightarrow (1)$

m_r be the inf of 'f' on $I_r = \frac{(r-1)a}{n}$

$F\left(\frac{r-1}{n}\right)a = \left(\frac{(r-1)a}{n}\right)^2 = m_r$

$L(P.f) = \sum_{r=1}^{n} m_r . \delta r$

$$= \sum_{r=1}^{n} \left(\frac{(r-1)a}{n}\right)^2$$

$$= \sum_{r=1}^{n} (r-1)^2 \frac{a^3}{n^3}$$

$$\Rightarrow \frac{a^3}{n^3} \sum_{r=1}^{n} (r-1)^2$$

$$\Rightarrow \frac{a^3}{n^3} \frac{(n-1)(n)(2n)}{6}$$

$$\Rightarrow \frac{a^3}{n^3} \frac{\cancel{2}n^3}{\cancel{6}_3}\left(1-\frac{1}{n}\right)$$

$$\Rightarrow \frac{a^3}{3}\left(1-\frac{1}{n}\right)$$

Now $\int_0^a f(x)\,dx = \lim_{n\to\infty} \frac{a^3}{3}\left(1-\frac{1}{n}\right)$

$$= \frac{a^3}{3}(1-0)$$

$$= \frac{a^3}{3} \rightarrow 2$$

From 1 & 2

$$\int_0^a f(x)\,dx = \frac{a^3}{3}$$

Theorem :- A constant function is Riemman integrable on $[a,b]$

proof :- Let $f(x)=K$, $\forall\, x \in [a,b]$ be a constant

Let $P = \{a = x_0, x_1, x_2 \cdots x_{r-1}, x_r, \cdots x_n = b\}$ be a partision on $[a,b]$

Let M_r, m_r be the sup and inf of 'f' on $I_r = [x_{r-1}, x_r]$

then $M_r = m_r = K$

now $U(P,f) = \sum^n M_r \delta_r$

$$= \sum_{r=1}^{n} k \cdot \delta_r$$

$$= k \sum_{r=1}^{n} \delta_r$$

$U(P,f) = k(b-a) \rightarrow (1)$

$L(P,f) = \sum_{r=1}^{n} k \cdot \delta_r$

$L(P,f) = k(b-a) \rightarrow (2)$

from (1) & (2)

$f(x)$ is riemman intergal on $[a,b]$

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} U(P,f)$$

$$= \lim_{n \to \infty} k(b-a)$$

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} L(P,f)$$

$$= \lim_{n \to \infty} k(b-a)$$

$$\overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx = k(b-a)$$

theorem :- If $f \in R[a,b]$ and m, M be a inf and sup of f on $[a,b]$ then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

proof :- We know that $\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx \rightarrow (1)$

Let $P = \{a = x_0, x_1, x_2, \cdots x_n = b\}$ be the partision on $[a,b]$

Suppose m, M be the inf & sup of f on $[a,b]$

Let M_r, m_r be the sup & inf of f on I_r

W.K.T $m \leq m_r \leq M_r \leq M$

$m\delta_r \leq m_r \delta_r \leq M_r \delta_r \leq M\delta_r$

$$\sum_{r=1}^{n} m\delta_r \le \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r \le \sum_{r=1}^{n} M\delta_r$$

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

Take $m(b-a) \le U(P,f)$

$\therefore m(b-a)$ is a lower bound of $\{U(P,f) / P \in \phi[a,b]\}$

$$m(b-a) \le \int_a^b f(x)dx.$$

$$m(b-a) \le \int_a^b f(x)\,dx \rightarrow ②$$

Take $L(P,f) \le M(b-a)$

$M(b-a)$ is upper bound of $\{L(P,f) / P \in \phi[a,b]\}$

$$M(b-a) \ge \sup\{L(P,f) / P \in \phi[a,b]\}$$

$$M(b-a) \ge \int_a^b f(x)\,dx$$

$$M(b-a) \ge \int_a^b f(x)\,dx \rightarrow ③$$

from ② & ③

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

NOTE: If $f:[a,b] \rightarrow R$ is a bounded Function and $P_1, P_2 \in \phi[a,b]$ such that $P_1 \subset P_2$ & P_2 contains n points more than P_1.

~~Darb~~

Darboux's theorem:

statement: If $f:[a,b] \to R$ is bounded Function then $\forall \epsilon > 0, \exists \delta > 0$ such that i) $U(P,f) < \overline{\int_a^b} f(x)dx + \epsilon$ ii) $L(P,f) > \underline{\int_a^b} f(x)dx - \epsilon$ for each partitian $P \in \wp[a,b]$ with $\|P\| < \delta$.

proof: suppose $f:[a,b] \to R$ is a bounded function $\exists K \in R$ such that $|f(x)| \leq K$, $x \in [a,b]$

i) Since $\overline{\int_a^b} f(x) dx = \inf \{U(P,f) / P \in \wp [a,b]\}$

$\overline{\int_a^b} f(x) dx + \frac{\epsilon}{2}$ is not lower bounded of $\{U(P,f)/P \in \wp (a,b]\}$

There is a partition P_1 such that $U(P,f) \leq \overline{\int_a^b} f(x) dx + \frac{\epsilon}{2} \to (1)$

Let P_1 have $(n-1)$ elements including the points a,b ~~closed~~ choose $\delta > 0$ such that $2K(n-1)\delta = \frac{\epsilon}{2}$

let P be a partition with $\|P\| < \delta$

Take $P_2 = P \cup P_1$

Then P_2 contains $(n-1)$ points more than 'P'.

$$U(P_1,f) - U(P_2,f) \leq 2K(n-1)\|P\|$$
$$< 2K(n-1)\delta$$

$$U(P_1,f) < U(P_2,f) + 2K(n-1)\delta \to (2)$$

since $P_1 \subset P_2$

$\therefore U(P_1,f) \geq U(P_2,f)$ (or) $U(P_2,f) \leq U(P_1,f) \to (3)$

from (2) & (3)

$$U(P,f) \leq U(P,f) + 2K(n-1)\delta$$
$$\leq \overline{\int_a^b} f(x)dx + \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad [\because \text{From } (1)]$$

$$U(P,f) = \overline{\int_a^b} f(x) dx + \epsilon$$

ii) $\int_a^b f(x)\,dx = \sup\{L(P,f) / P \in \wp[a,b]\}$

$\int_{\underline{a}}^{b} f(x)\,dx - \frac{\epsilon}{2}$ is not upper bound of $\{L(P,f) / P \in \wp(a,b)\}$

There is a partision P such that $L(P,f) > \int_{\underline{a}}^{b} f(x)\,dx$

Let P_1 have $(n-1)$ elements excluding the points a, b.

chosse $\delta > 0$ such that $2k(n-1)\delta = \frac{\epsilon}{2}$

Let P be a partisian with $\|P\| = \delta$

Take $P_2 = P \cup P_1$

Then P_2 containg almost $(n-1)$ points more than 'p'.

$L(P_2, f) - L(P_1, f) \leq 2k(n-1)\|P\| \leq 2k(n-1)\delta$

$-L(P_1, f) < 2k(n-1)\delta - L(P_2, f)$

$L(P, f) > L(P_2, f) - 2k(n-1)\delta \rightarrow (5)$

$\therefore P_1 \subset P_2 \quad L(P_1, f) \geq L(P_2, f)$

(or)

$L(P_2, f) \leq L(P_1, f) \rightarrow (6)$

From (5) & (6)

$L(P_1, f) > L(P, f) > L(P, f) - 2k(n-1)\delta \geq \int_{\underline{a}}^{b} f(x)\,dx - \frac{\epsilon}{2} - \frac{\epsilon}{2}$

$L(P, f) \geq \int_{\underline{a}}^{b} f(x)\,dx - \epsilon$

Borell's theorem :

statement : If a function 'f' is continuous on $[a,b]$ then for each $\epsilon < 0$ there exist a partisian P on $[a,b]$ such that $|f(x_1) - f(x_2)| < \epsilon, \forall x \in [a,b]$ and $x_1, x_2 \in I_x$.

Theorem (6) : Imp

A function $f : (a,b] \rightarrow R$ is continuous on $[a,b]$ is Riemann integrable

Proof : Suppose 'f' is continuous on $[a,b]$

Let $E > 0 \Rightarrow \frac{E}{b-a} > 0$

By Borell's theorem if for each $\frac{E}{b-a} > 0$ ∃ a partision.

$\therefore P \in \wp [a,b]$ such that $|f(x_1) - f(x_2)| < \frac{E}{b-a} \; \forall x_1, x_2 \in I_r \rightarrow (1)$

'f' is also attains it's bounds on $[a,b]$

Let $\alpha_r, \beta_r \in I_r$

$f(\alpha_r) = M_r$ & $f(\beta_r) = m_r$

Now, $M_r - m_r = |f(\alpha_r) - f(\beta_r)| < \frac{E}{b-a}$

$$U(P_1, f) - L(P_1, f) = \sum_{r=1}^{n} M_r \delta_r - \sum_{r=1}^{n} m_r \delta_r$$

$$= \sum_{r=1}^{n} (M_r - m_r) \delta_r$$

$$\therefore U(P_1, f) - L(P_1, f) < \sum_{r=1}^{n} (\delta_1) \frac{E}{b-a}$$

$$< (b-a) \frac{E}{b-a} \quad \left[\because \sum_{r=1}^{n} \delta_r = b-a\right]$$

$$U(P_1, f) - L(P_1, f) < E$$

$\therefore$ 'f' is Riemann integral.

Monotonic Function :

A Function which is eithe r increasing or decreasing is called Monotonic Function.

Theorem - 7 :

If $f : [a,b] \rightarrow R$ is monotonic function on $[a,b]$ then 'f' is Riemann integral.

Proof : suppose 'f' is Monotonic Function on $[a,b] \Rightarrow$ 'f' is increasing (or) decreasing Function.

Now, We take 'f' is increasing Function.

$\therefore$ 'f' is increasing Function $\Leftrightarrow a \le x \le b$

$\Leftrightarrow f(a) \le f(x) \le f(b)$

Let sup of 'f' on $[a,b] = f(b)$ and inf of 'f' on $[a,b] = f(a)$ such that $\epsilon > 0$ and a Partision.

$P = \{a = x_0, x_1 \ldots\ldots x_n = b\}$ on $[a,b]$

let $\delta_r = \dfrac{\epsilon}{f(b) - f(a) + 1}$. Where, $r = 1, 2, \ldots\ldots n$.

Let M_r, m_r be the sup and inf of 'f' on $I_r = [x_r - 1]$

Then, $M_r = f(x_r)$ $m_r = f(x_r - 1)$

Now, $U(P_1, f) - L(P_1, f) = \sum_{r=1}^{n} M_r \delta_r - \sum_{r=1}^{n} m_r \delta_r$

$$= \sum_{r=1}^{n} \delta_r (M_r - m_r)$$

$$U(P_1, f) - L(P_1, f) < \sum_{i=1}^{n} (M_r - m_r) \frac{\epsilon}{f(b) - f(a) + 1}$$

$$< \left[f(x_1) - f(x_0) + f(x_2) - f(x_1) + \ldots\ldots + f(x_n) - f(x_n - 1) \frac{\epsilon}{f(b) - f(a) + 1}\right]$$

$$< \left[f(x_n) - f(x_0)\right] \frac{\epsilon}{f(b) - f(a) + 1}$$

$$< \left[f(b) - f(a)\right] \frac{\epsilon}{f(b) - f(a) + 1}$$

$$< \epsilon$$

$U(P_1, f) - L(P_1, f) < \epsilon$

$\therefore$ 'f' is riemann integrable.

Statement: If $f \in R[a,b]$ and M, m are sup & Inf of 'f' on $[a,b]$ then $m(b-a) < \int_a^b f(x)\,dx < M(b-a)$ & $\int_a^b f(x)\,dx = \mu(b-a)$ Where $\mu \in (m, M)$

Proof: suppose $f \in R[a,b]$ and m, M are sup & Inf of f on $[a,b]$

We know that, $m \leq f(x) \leq M$.

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M(dx)$$

$$m[x]_a^b \leq \int_a^b f(x)\,dx \leq M[x]_a^b$$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \rightarrow ①$$

divide ① by $(b-a)$

$$m \leq \int_a^b \frac{f(x)}{b-a}\,dx \leq M \rightarrow ②$$

Take $\int_a^b \frac{f(x)}{b-a}\,dx = \mu$

From Equ ② $m \leq \mu \leq M$

$$\int_a^b f(x)\,dx = \mu(b-a) \text{ Where } \mu \in (m, M)$$

Theorem:

Statement: A bounded Function $f : [a,b] \rightarrow R$ is a Riemann Integrable iff for each $\epsilon > 0$ there exists a partisian P of $[a,b]$ such that $0 \leq U(P,f) - L(P,f) < \epsilon$

Proof: suppose f is Riemann integrable on $[a,b]$.

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \rightarrow ①$$

Let $\epsilon > 0 \Rightarrow \frac{\epsilon}{2} > 0$

By Darboux's theorem,

$$U(P_1,f) < \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} \text{ and } L(P,f) > \int_a^b f(x)\,dx - \frac{\epsilon}{2} \rightarrow (2)$$

From equ (1);

$$U(P,f) < \int_a^b f(x)\,dx + \frac{\epsilon}{2} \text{ and } L(P,f) > \int_a^b f(x)\,dx - \frac{\epsilon}{2}$$

$$L(P,f) + \frac{\epsilon}{2} > \int_a^b f(x)\,dx \ (\text{a})$$

$$\int_a^b f(x)\,dx < L(P,f) + \frac{\epsilon}{2} \rightarrow (3)$$

$$U(P,f) < \int_a^b f(x)\,dx + \frac{\epsilon}{2}$$

$$< L(P,f) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad (\because \text{From } (3))$$

$$U(P,f) < L(P,f) + \epsilon$$

$$U(P,f) < L(P,f) < \epsilon \rightarrow (4)$$

$$\text{WKT}, U(P,f) - L(P,f) \geq 0$$

A Necessary and sufficient condition for integrability.

$$(\text{or}) \quad 0 \leq U(P,f) - L(P,f) \rightarrow (5)$$

from (4) & (5)

$$\Rightarrow 0 \leq U(P,f) - L(P,f) < \epsilon$$

Now, we P.T 'f' is Riemann integrable

By definition $\overline{\int_a^b} f(x)\,dx = \text{Inf}\{U(P,f) \mid P \in \phi[a,b]\}$

$$\overline{\int_a^b} f(x)\,dx \leq U(P,f) \rightarrow (6)$$

By definition

$$\overline{\int_a^b} f(x)\,dx - \sup\{L(P,f) \mid P \in \phi[a,b]\}$$

$$\underline{\int_a^b} f(x)\,dx \geq L(P,f) \longrightarrow (7)$$

From (6) & (7)

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq U(P,f) - L(P,f) < \epsilon$$

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \epsilon \longrightarrow (8)$$

W.K.T

$$\overline{\int_a^b} f(x)\,dx \geq \underline{\int_a^b} f(x)\,dx$$

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \geq 0 \quad (a)$$

$$0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \longrightarrow (9)$$

From (8) & (9)

$$0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx$$

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx$$

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

$\therefore$ 'f' is reimann integrable.